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MAGIC POLYHEDRONS

Close your eyes and imagine that you are connecting the midpoint of a cube with its vertices by line segments, creating in this way six congruent square pyramids, which completely fill this cube.

Now duplicate each of these pyramids by reflecting each of them on the plane given by its base. You get now 6 square pyramids positioned onto the faces of the cube outside. The cube together with these six pyramids perform a new polyhedron.

Draw this polyhedron in that way you can imagine it. Then answer the following questions:

- How many (genuine) faces/ edges/ vertices does have this new polyhedron?
- Which kind of polygonal shapes are its faces of?
- Are its faces congruent?
- Is this polyhedron a regular one?
- What's its volume? (Compare the volume of this polyhedron with the volume of the cube in regard with the method you did create it.)

The new solid is a polyhedron with 14 vertices. 8 of them belong to the cube, and the remaining 6 you did create additionally. The polyhedron has 24 edges, none of them, as you can see, belongs to the cube.

The edges of the base cube are now integrated into the faces of the new polyhedron and they are identical with the shorter diagonals of the rhombuses.

Of course, the polyhedron is not regular although the faces are congruent, but it does have incongruent corners: either three or four rhombuses built a corner.

From number and shape of its faces the name of the new polyhedron is derived:

rhombus dodecahedron.

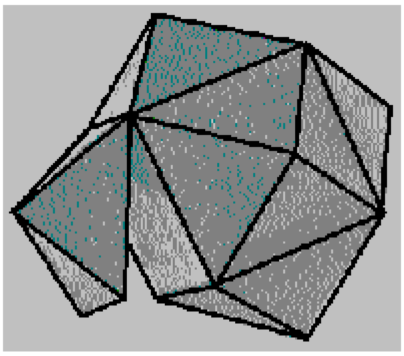
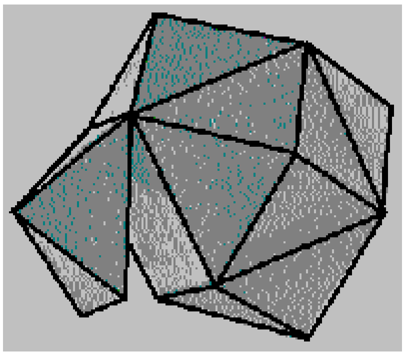


fig 1

After creation of the rhombus dodecahedron in our imagination we use now materialized models for fitting the 6 suitable square pyramides to the sides of a cube using adhesive tape. Then we cut them partially off along the edges with a knife or a razor blade and we get a model enabling us to visualize the fact that the volume of the rhombus dodecahedron is twice the volume of the cube from which it was generated.

Below we illustrate the phases of transforming the rhombus dodecahedron into two cubes with a common side. First, we wind out two pyramids from the opposite sides of the cube which is located inside the dodecahedron.....



{ EINBETTEN Word.Picture.8 }

fig 2a/b

...Then we detach succesively the remaining pyramids and put them one by one in the line....

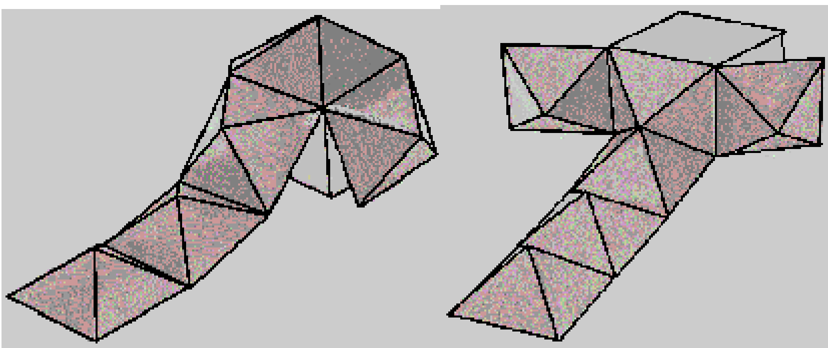


fig 3a/b

.... Finally we fit them together to the second cube...

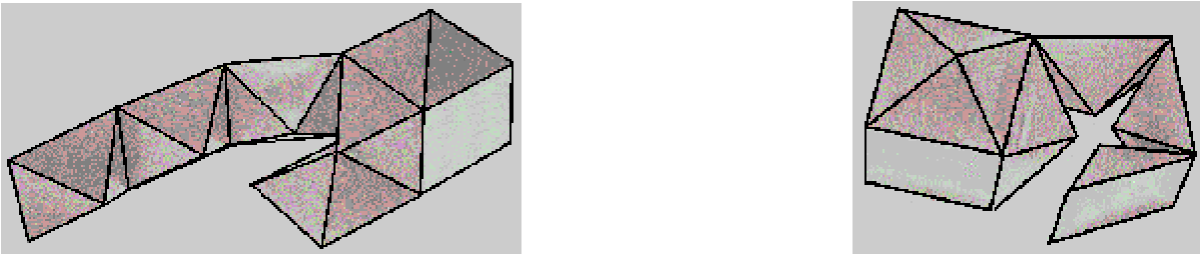


fig 4a/b

fig 5a/b

It is easy to recognize that the edge of the rhombus dodecahedron is one half of a spatial diagonal line of the cube. This means that for the edge *„a"* of the cube, the edge of the rhombus dodecahedron is *b* = **{** EINBETTEN Equation.3 **}**. This gives to us an easy de-rivation of the formula for the volume of the rhombus dodecahedron with the edge b:

{ EINBETTEN Equation.3 **}**

The reader is able to calculate in the mind the surface of the rhombus dodecahedron with the side *„b"*.

The fact that one rhombus dodecahedron with the edge *„b"* may be transformed into two cubes of the edge *„a"* which allows us to make further deductions.
From eight cubes, each of the edge *a* we may create another cube with an edge of double length (because $2^3 = 8$). Based on the preceeding reasoning, the eight cubes may be transformed into four rhombus dodecahedron. That means that into the cube of the edge *„2a"*, we may pack four rhombus dodecahedrons of the edge *b* = **{** EINBETTEN Equation.3 **}**. But how to do that? The easiest way would be to make reverse tranformation into the cubes and square pyramides, however it is not as convenient as it might be supposed.

Let's try something different. First put one of such dodecahedron into the cube of the double length edge of the cube, from which it was created. Recognize, that the distance between the opposite situated vertices of the dodecahedron is the height of the cube into which we are going to fill in. This suggests how it should be done
- fig. 6.

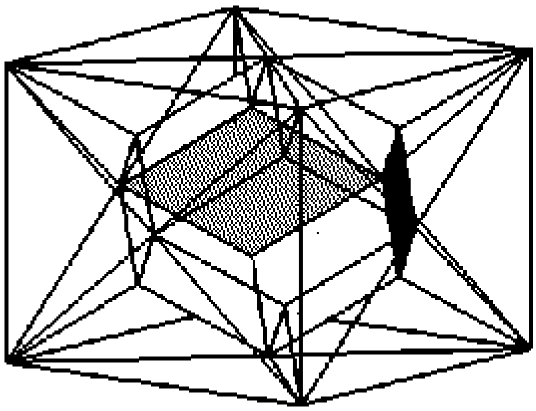


fig 6

We are aware that the dodecahedron dissected into four congruent heptahedrons could be postioned onto the bottom of the cube. One of its edges equals the cube's edge with the length *„2a"*.

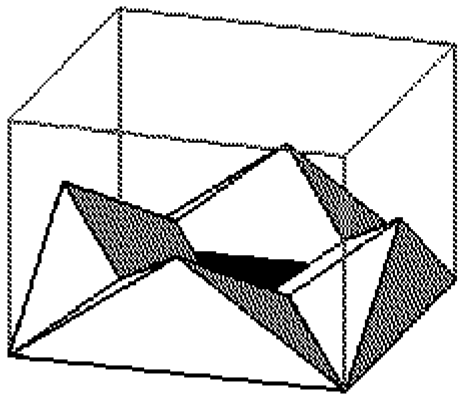


fig 7

Now is clear that to each of the heptahedrons belong:

- four faces which are isosceles triangles with the base of length of **{** EINBETTEN Equation.2 **}**, side of length of **{** EINBETTEN Equation.2 **}**/2 and the height of *a/2*,
- the rhombus with the length diagonal **{** EINBETTEN Equation.2 **}** and *a* ,
- two rectangular isosceles triangles with the hypotenuse *2a* and the leg of right angle with the length **{** EINBETTEN Equation.2 **}.**

It's interesting, that four such heptahedrons create one rhombus dodecahedron, the same one, that has been put into the cube (fig 10).

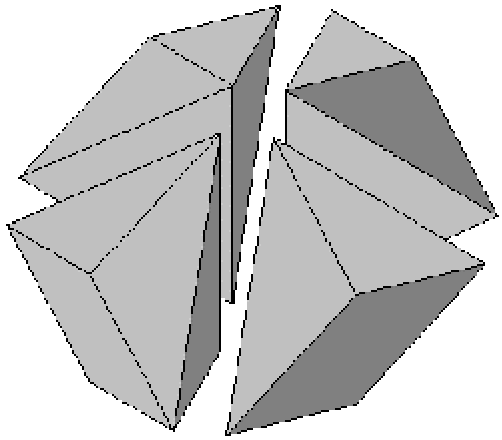


fig 8

Do you know how many of these heptahedrons you can place in the cube with the edge „*2a”*? Try to figure it out: the volume of this cube is $8a^3$, so you can place inside 8 cubes with the edge „a”. Each couple of them does have the same volume as one adequate dodecahedron. It means, that inside the cube which you are just beginning to fill you can place four rhombus dodecahedrons, that are exactly sixteen heptahedrons.

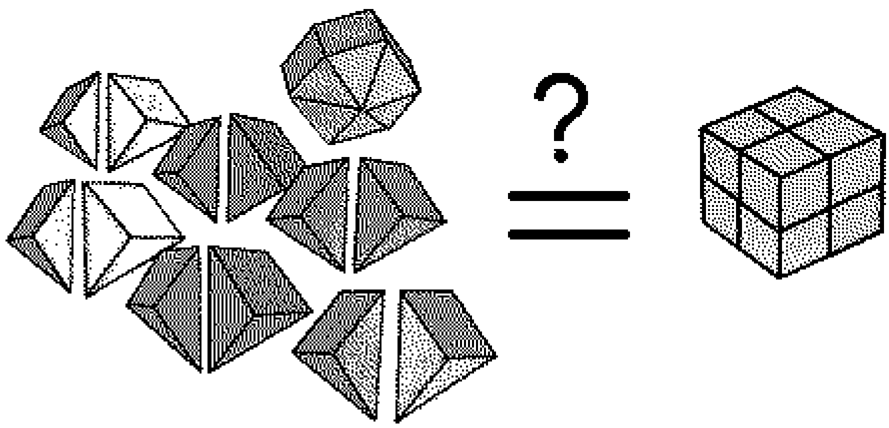


fig 9

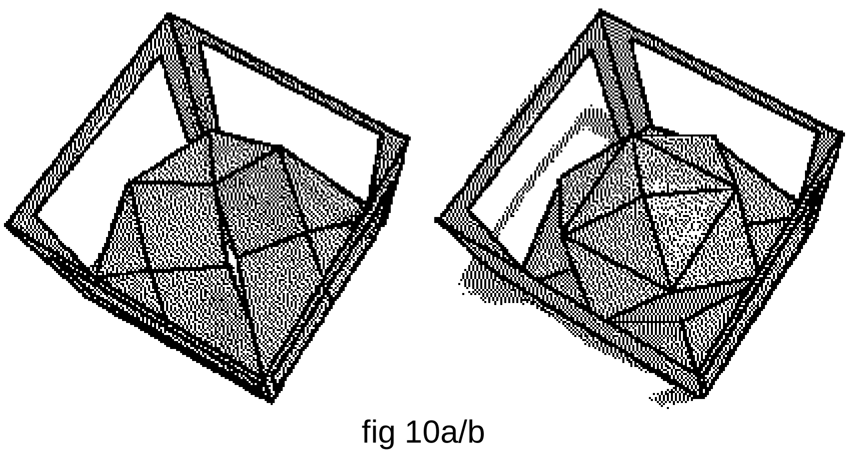


fig 10a/b

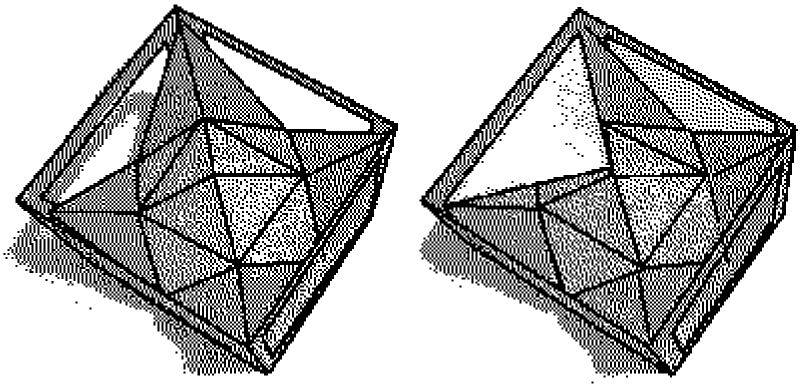


fig 11a/b

At last we show the net of the "magic" heptahedron including its measurement:

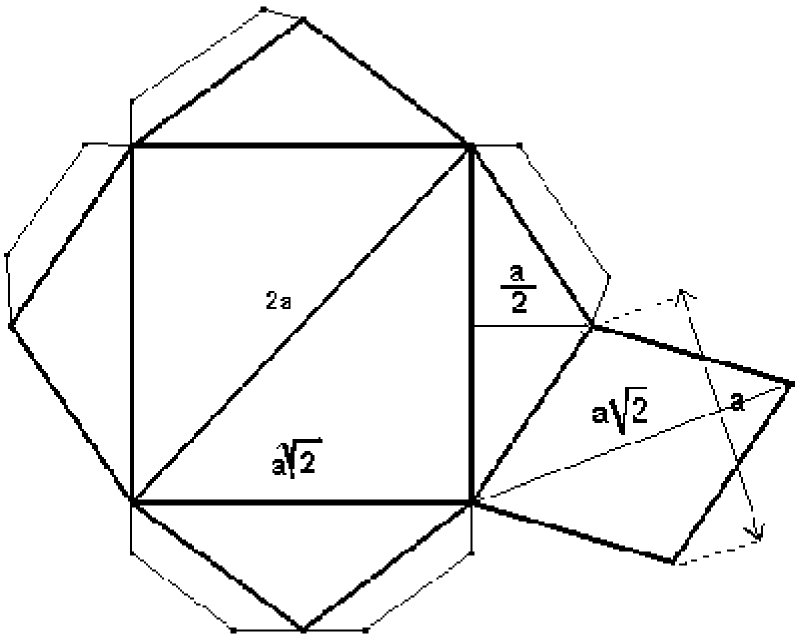


fig 12